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Semiclassical Gravitoelectromagnetic inflation in a Lorentz gauge: seminal inflaton fluctuations and electromagnetic fields from a 5D vacuum state

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Using a semiclassical approach to Gravitoelectromagnetic Inflation (GEMI), we study the origin and evolution of seminal inflaton and electromagnetic fields in the early inflationary universe from a 5D vacuum state. The difference with other previous works is that in this one we use a Lorentz gauge. Our formalism is naturally not conformal invariant on the effective 4D de Sitter metric, which make possible the super adiabatic amplification of magnetic field modes during the early inflationary epoch of the universe on cosmological scales.

I. INTRODUCTION

The origin of cosmological scales magnetic fields is one of the most important, fascinating and challenging problems in modern cosmology. Many scenarios have been proposed to explain them. Magnetic fields are known to be present on various scales of the universe[1]. Primordial large-scale magnetic fields may be present and serve as seeds for the magnetic fields in galaxies and clusters.

Until recently the most accepted idea for the formation of large-scale magnetic fields was the exponentiation of a seed field as suggested by Zeldovich and collaborators long time ago. This seed mechanism is known as galactic dynamo. However, recent observations have cast serious doubts on this possibility. There are many reasons to believe that this mechanism cannot be universal. This is why the mechanism responsible for the origin of large-scale magnetic fields is looked in the early universe, more precisely during inflation[2], which should be amplified through the dynamo mechanism after galaxy formation. In principle, one should be able to follow the evolution of magnetic fields from their creation as seed fields through to dynamo phase characteristic of galaxies. It is believed that magnetic fields can play an important role in the formation and evolution of galaxies and their clusters, but are probably not essential to our understanding of large-scale structure in the universe. However, an understanding of structure formation is paramount to the problem of galactic and extragalactic magnetic fields[3, 4].

It is natural to look for the possibility of generating large-scales magnetic fields during inflation with strength according with observational data on cosmological scales: $< 10^{-9}$ Gauss. However, the FRW universe is conformal flat and the Maxwell theory is conformal invariant, so that magnetic fields generated at inflation would come vanishingly small at the end of the inflationary epoch. The possibility to solve this problem relies in produce non-trivial magnetic fields in which conformal invariance to be broken.

On the other hand, the five dimensional model is the simplest extension of General Relativity (GR), and is widely regarded as the low-energy limit of models with higher dimensions (such as 10D supersymmetry and 11D supergravity). Modern versions of 5D GR abandon the cylinder and compactification conditions used in original Kaluza-Klein (KK) theories, which caused problems with the cosmological constant and the masses of particles, and consider a large extra dimension. In particular, the Induced Matter Theory (IMT) is based on the assumption that ordinary matter and physical fields that we can observe in our 4D universe can be geometrically induced from a 5D Ricci-flat metric with a space-like noncompact extra dimension on which we define a physical vacuum[5].

Gravitoelectromagnetic Inflation (GEMI)[6] was proposed recently with the aim to describe, in an unified manner, electromagnetic, gravitational and the inflaton fields in the early inflationary universe, from a 5D vacuum. It is known that conformal invariance must be broken to generate non-trivial magnetic fields. A very important fact is that in this formalism conformal invariance is naturally broken. Other conformal symmetry breaking mechanisms have been proposed so far[7]. In the framework of the IMT, electromagnetic effects were studied at a classical level in[8]. However, most of these are developed in the Coulomb gauge. In this paper we study a semiclassical approach of this formalism in a Lorentz gauge. Furthermore, for simplicity we shall neglect back reaction effects on the semiclassical Einstein equations.

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II. VECTOR FIELDS IN 5D VACUUM

We begin considering a 5D manifold $\mathcal{M}$ described by a symmetric metric $g_{ab} = g_{ba}$[1]. This manifold $\mathcal{M}$ is mapped by coordinates $\{x^a\}$

$$dS^2 = g_{ab} dx^a dx^b, \tag{1}$$

where g_{ab} is the 5D tensor metric, such that $g_{ab} = g_{ba}$. From the geometrical point of view, to describe a relativistic 5D vacuum, we shall consider that g_{ab} is such that the Ricci tensor $R_{ab} = 0$, and hence: $G_{ab} = 0$. To describe the system we introduce the action on the manifold $\mathcal{M}$

$$\mathcal{S} = \int d^5x \sqrt{-g} \left[\frac{{}^{(5)}R}{16\pi G} - \frac{1}{4} Q_{bc} Q^{bc} \right], \tag{2}$$

where ${}^{(5)}R$ is the 5D scalar curvature on the five-dimensional metric (1) and $Q^{ab} = F^{ab} - \gamma g^{ab} \nabla_f A^f$, where the 5D Faraday tensor is $F^{bc} = \nabla^b A^c - \nabla^c A^b = \partial^b A^c - \partial^c A^b$. We shall consider that the fields A^b are minimally coupled to gravity and free of interactions, so that the second term in the action is purely kinetic.

A. Einstein Equations in 5D

If we minimize the action respect to the metric we will obtain Einstein Equations in 5D. In this paper we shall use a semiclassical approach where the Einstein equations are expressed by the homogeneous component of the fields. This slightly differs from the one used by [9] in the fact that we don't need to renormalize the stress tensor, but at the cost of assuming a semiclassical behavior of the fields that rules out the dependence with the wavenumber in the calculation of the semiclassical Einstein equations

$$G_{ab} = -8\pi G\, T^{(0)}_{ab}, \tag{3}$$

where $T^{(0)}_{ab} \equiv \langle T_{ab}(\bar{A}^c) \rangle$. Notice that we use a semiclassical expansion of the vector fields

$$A^c = \bar{A}^c + \delta A^c, \tag{4}$$

where the overbar symbolizes the 3D spatially homogeneous background field consistent with the fixed homogeneous metric and δA^c describes the fluctuations with respect to $\bar{A}^c$. In this sense when we perform the expectation value of the stress tensor, adopting the ansatz $\langle \delta A_c \rangle = 0$, only will appear zero order $T^{(0)}_{ab}$ and the second order $T^{(2)}_{ab}$ in perturbations terms. The last corresponds to a feedback term and is related to back-reaction effects, which do not will be consider in this paper. The stress tensor is defined by the fields lagrangian being symmetric by definition

$$T_{bc} = \frac{2}{\sqrt{g}} \left\{ \frac{\partial}{\partial g^{bc}} \left(\sqrt{g} \mathcal{L}_f \right) - \frac{\partial}{\partial x^e} \left[\frac{\partial}{\partial {g^{bc}}_{,\,e}} \left(\sqrt{g} \mathcal{L}_f \right) \right] \right\}. \tag{5}$$

The appearance of variations respect to derivatives of the metric is because we are dealing with vector fields whose covariant derivative operators involve Christoffel symbols (i.e. ordinary derivatives of the metric).

In our case the stress tensor reduces to

$$\begin{aligned} T_{bc} &= {F^e}_b F_{ce} + \frac{1}{4} g_{bc} F_{de} F^{de} - \lambda \left\{ 2{A^e}_{;\,e} \left[A_{(b;\,c)} - \left(2A_{(b} g_{c)h,\,f} + g_{hf,\,(b} A_{a)} \right) g^{hf} \right] + \right. && (6)\\ &+ g_{bc} \left[\left(A^e_{,\,ef} + \Gamma^e_{de,\,f} A^d + \Gamma^e_{de} A^d_{,\,f} + 2\Gamma^e_{ef} A^d_{,\,d} + \frac{3}{2} \Gamma^e_{ed} \Gamma^a_{af} A^d \right) A^f + \frac{1}{2} \left(A^e_{,\,e} \right)^2 \right] \\ &+ \left. g_{bc,\,f} A^f A^e_{;\,c} \right\}, && (7) \end{aligned}$$

where $\gamma^2 = \frac{2\lambda}{5}$.

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[1] In our conventions latin indices "a,b,c,..,h" run from 0 to 4, greek indices run from 0 to 3 and latin indices "i,j,k,..." run from 1 to 3.

B. 5D dynamics of the fields

The Euler-Lagrange equations give us the dynamics for A_b

$$\nabla_f \nabla^f A^b - R^b_f A^f - (1-\lambda)\nabla^b \nabla_f A^f = 0. \tag{8}$$

In particular, the choice $\lambda = 1$ is known as Feynman gauge, somehow equivalent to a Lorentz gauge $\nabla_f A^f = 0$. In this paper we shall choose simultaneously both conditions. It is easy to show that the 5-divergence of the field equation of motions satisfy the same equation as in a Minkowski space, but changing ordinary partial derivatives by the covariant derivative

$$\nabla^a \nabla_a \left(\nabla_f A^f\right) = 0. \tag{9}$$

Hence, the Lorentz gauge is satisfied for appropriate initial conditions of $\nabla_a A^a = 0$. With such a choice the field lagrangian density $\mathcal{L}_f = -\frac{1}{4}Q^2$ is

$$\mathcal{L}'_f = -\frac{1}{2}\nabla_a A_b \nabla^a A^b = -\frac{1}{2}\nabla_\mu A_\nu \nabla^\mu A^\nu - \frac{1}{2}\nabla_4 A_\nu \nabla^4 A^\nu - \frac{1}{2}\nabla_\mu A_4 \nabla^\mu A^4 - \frac{1}{2}\nabla_4 A_4 \nabla^4 A^4. \tag{10}$$

For 4D observers living in a hypersurface where the fifth component of the vector field is normal to it, this extra dimensional field will manifest separately, like an effective 4D vector field A^ν and a 4D scalar field A^4. In this sense we can identify kinetic terms for both, scalar and vector fields, and the derivatives with respect to the extra dimension may be interpreted as potential (or dynamical sources) terms joined with massive terms for each of them.

The stress tensor in this gauge is

$$\begin{aligned} T_{ab} = \quad & -\nabla_a A_e \nabla_b A^e - \nabla_e A_a \nabla^e A_b - 2g_{c(b}A_{a)}\Gamma^c_{ef}\nabla^e A^f + \frac{1}{2}g_{ab}\nabla_e A_f \nabla^e A^f - \\ & 2\frac{g_{,f}}{g}\left[\nabla_{(a}A_{b)}A^f + \nabla^f A_{(b}A_{a)} - A_{(a}\nabla_{b)}A^f\right] - \left[\nabla_{(a}A_{b)}A^f + \nabla^f A_{(b}A_{a)} - A_{(a}\nabla_{b)}A^f\right]_{,f}. \end{aligned} \tag{11}$$

III. SPECIAL CASE: 5D GENERALIZATION OF A DE SITTER SPACETIME

Because we are interested to study a cosmological scenario of inflation from the context of the theory of Space-Time-Matter, we shall consider the 5D Riemann-flat metric[10]

$$dS^2 = \psi^2 dN^2 - \psi^2 e^{2N} dr^2 - d\psi^2, \tag{12}$$

where N is a time-like dimension related to the number of e-folds, $dr^2 = dx^i \delta_{ij} dx^j$ is the Euclidean line element in cartesian coordinates and ψ is the space-like extra dimension. This metric satisfies the vacuum condition $G^{ab} = 0$.

For this 5D metric the field equations, after taking Lorentz gauge: $\nabla_a A^a = \partial_N A^0 + 3A^0 + \partial_\psi A^4 + 4\psi^{-1}A^4 + \partial_i A^i = 0$, are

$$\left\{\frac{\partial^2}{\partial N^2} + 5\frac{\partial}{\partial N} - e^{-2N}\partial_r^2 - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi}\right]\right\} A^0 + \left[\frac{2}{\psi}\frac{\partial}{\partial N} + 2\frac{\partial}{\partial \psi} + \frac{8}{\psi}\right] A^4 = 0, \tag{13}$$

$$\left\{\frac{\partial^2}{\partial N^2} + 5\frac{\partial}{\partial N} - e^{-2N}\partial_r^2 - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi}\right]\right\} A^j - 2\partial^j\left(A^0 + \frac{A^4}{\psi}\right) = 0, \tag{14}$$

$$\left\{\frac{\partial^2}{\partial N^2} + 3\frac{\partial}{\partial N} - e^{-2N}\partial_r^2 - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi} + \frac{12}{\psi^2}\right]\right\} A^4 = 0. \tag{15}$$

Notice that the (15) is decoupled after applying the Lorentz gauge. However we see that it is not sufficient to decouple all the field equations. This is because the non zero connections of the metric (12) act in a non trivial manner in the vector fields derivatives. There are 14 non zero Christoffel symbols

$$\Gamma^\mu_{\mu 4} = \psi^{-1}, \quad \Gamma^i_{i0} = 1, \quad \Gamma^0_{ii} = e^{2N}, \quad \Gamma^4_{00} = \psi, \quad \Gamma^4_{ii} = -\psi e^{2N}. \tag{16}$$

Therefore, in this Riemann-flat spacetime we obtain the D 'Alambertian of the A^b field

$$\nabla_f \nabla^f A^b = 0, \tag{17}$$

but, expressed in terms of the ordinary derivatives and the Christoffel symbols we notice the coupling terms

$$g^{fh}\left\{\partial_f\partial_h A^b + 2\Gamma^b_{ef}\partial_h A^e + \Gamma^b_{he,\,f}A^e - \Gamma^e_{fh}\partial_e A^b - \Gamma^e_{fh}\Gamma^b_{ed}A^d + \Gamma^b_{ef}\Gamma^e_{hd}A^d\right\} = 0. \tag{18}$$

In the Minkowskian limit ($\psi_0 \to \infty$) all of the connections vanish and so the field equations remain decoupled after the gauge choice.

A. Dynamics of the 3D spatially isotropic background fields

We shall combine the field equations of motion for the classical homogeneous fields with the Einstein Equations, the first ones reduce to

$$\left\{\frac{\partial^2}{\partial N^2} + 5\frac{\partial}{\partial N} - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi}\right]\right\}\ \bar{A}^0\ + \left[\frac{2}{\psi}\frac{\partial}{\partial N} + 2\frac{\partial}{\partial \psi} + \frac{8}{\psi}\right]\bar{A}^4 = 0, \tag{19}$$

$$\left\{\frac{\partial^2}{\partial N^2} + 5\frac{\partial}{\partial N} - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi}\right]\right\}\ \bar{A}^j\ = 0, \tag{20}$$

$$\left\{\frac{\partial^2}{\partial N^2} + 3\frac{\partial}{\partial N} - \psi^2\left[\frac{\partial^2}{\partial \psi^2} + \frac{6}{\psi}\frac{\partial}{\partial \psi} + \frac{12}{\psi^2}\right]\right\}\ \bar{A}^4\ = 0. \tag{21}$$

Notice that the equation for $\bar{A}^0$ is the unique coupled. Furthermore, once obtained $\bar{A}^4$, we can describe the dynamics of $\bar{A}^0$ in (19), where $\bar{A}^4$ appears as a source.

IV. EFFECTIVE 4D DYNAMICS OF THE FIELDS

The remarkable property of the 5D metric (12) is that it is a generator of 4D de Sitter spacetimes. This may be done when we foliate the space (12) in a particular hypersurface $\psi = \psi_0$. It is said that for an observer moving with the penta velocity $U_\psi = 0$, the spacetime describes a de Sitter expansion. Then the effective 4D hypersurface it has a scalar curvature ${}^{(4)}R = 12/\psi_0^2 = 12\,H_0^2$, such that the Hubble parameter is defined by the foliation $H_0 = \psi_0^{-1}$. If we consider the coordinate transformations

$$t = \psi_0 N, \quad R = \psi_0 r, \quad \psi = \psi, \tag{22}$$

we then arrive to the Ponce Leon metric[11]: $dS^2 = \left(\frac{\psi}{\psi_0}\right)^2\left[dt^2 - e^{2t/\psi_0}dR^2\right] - d\psi^2$. If we foliate $\psi = \psi_0$, we get the effective 4D metric

$$dS^2 \to ds^2 = dt^2 - e^{2H_0 t}d\vec{R}^2, \tag{23}$$

which describes a 3D spatially flat, isotropic and homogeneous de Sitter expanding universe with a constant Hubble parameter H_0.

The dynamics of the fields being given by the equations (13), (14) and (15), evaluated on the foliation $\psi = \psi_0 = 1/H_0$, with the transformations (22). In the following subsections we shall study separately the dynamics of the classical 3D spatially isotropic fields: $\bar{A}^\mu(t,\psi_0)$ and $\bar{A}^4(t,\psi_0)$, and the fluctuations of these fields: $\delta A^\mu(t,\vec{R},\psi_0)$ and $\delta A^4(t,\vec{R},\psi_0)$. Notice that now $\vec{R} \equiv \vec{R}(X^i)$. To describe the dynamics of the fields we shall impose the effective 4D Lorentz gauge: ${}^{(4)}\nabla_\mu A^\mu = 0$. It implies that the 5D Lorentz gauge with the transformations (22) and evaluated on the foliation must now be

$$\nabla_a A^a\big|_{\psi_0} =^{(4)} \nabla_\mu A^\mu(t,\vec{R},\psi_0) + \left(\partial_\psi A^4 + 4\psi^{-1}A^4\right)\big|_{\psi_0} = 0, \tag{24}$$

where ${}^{(4)}\nabla_\mu\, A^\mu$ denotes the covariant derivative on the effective 4D metric (23). Hence, in order to the effective 4D Lorentz gauge to be fulfilled, we shall require

$$\left(\partial_\psi A^4 + 4\psi^{-1}A^4\right)\big|_{\psi_0} = 0. \tag{25}$$

A. 4D classical field dynamics

In order to solve the equations (19), (20) and (21) on an effective 4D de Sitter metric, we must evaluate these equations on the particular foliation $\psi=\psi_0=H_0^{-1}$, $r=R\,\psi_0$ and $N=H_0\,t$. We shall identify the effective scalar A^4 with the inflaton field: $A^4(t,\vec{R},\psi_0)\equiv\phi(t,\vec{R},\psi_0)$ and we shall denote $\bar{\phi}(t,\psi_0)\sim\phi_1(N)\,\phi_2(\psi)|_{N=H_0t,\psi=\psi_0=H_0^{-1}}$, as the 3D spatially isotropic and homogeneous background field. In the same way we state for the homogeneous component of the vector field the separation $\bar{A}^j(t,\psi_0)\sim S_1^j(N)S_2^j(\psi)\Big|_{N=H_0t,\psi=\psi_0=H_0^{-1}}$, in the next we shall drop the index j to label the functions $S_1(t)$ and $S_2(\psi_0)$. Hence, we obtain

$$\bar{\phi}(t,\psi_0)\sim\phi_m(t,y_0)=e^{-\frac{3}{2}H_0t}\left(a_1\,e^{\alpha H_0t}+a_2\,e^{-\alpha H_0t}\right),\qquad\alpha=\frac{3}{2}\sqrt{1-\frac{4m^2}{9}},\tag{26}$$

where we have considered the condition (25), such that

$$-\psi^2\left[\frac{\partial^2}{\partial\psi^2}+\frac{3}{\psi}\frac{\partial}{\partial\psi}\right]\bar{A}^4\Big|_{\psi_0}=m^2\,\bar{\phi}(t,\psi_0).\tag{27}$$

Furthermore, the general solution of eq. (20) on the effective 4D metric (23), is

$$\bar{A}^j(t,\psi_0)\sim S_\mu(t)=e^{-\frac{5}{2}H_0t}\left(c_1\,e^{\gamma H_0t}+c_2\,e^{-\gamma H_0t}\right),\qquad\gamma=\frac{5}{2}\sqrt{1-\frac{4\nu^2}{25}}\tag{28}$$

where

$$-\psi^2\left[\frac{\partial^2}{\partial\psi^2}+\frac{6}{\psi}\frac{\partial}{\partial\psi}\right]\bar{A}^j\Big|_{\psi_0}=\nu^2\,\bar{A}^j(t,\psi_0).\tag{29}$$

A similar treatment can be done for $\bar{A}^0$, after making use of the condition (25), the transformations (22) and the foliation $\psi=\psi_0=1/H_0$. However, the difference with the other background components of the field observed in eq. (19) is that $\bar{A}^4\equiv\bar{\phi}(t,\psi_0)$ acts as a source of $\bar{A}^0(t,\psi_0)$.

As a particular choice we shall consider a 4D inflationary universe, where the background fields are $\bar{A}^b=\left(0,0,0,0,\bar{\phi}\right)$, in agreement with a global (de Sitter) accelerated expansion which is 3D spatially isotropic, flat and homogeneous[2]. In this case, the relevant components of the classical Energy momentum tensor, are

$$\begin{aligned}\rho\equiv\langle T_0^0\rangle&=\frac{1}{2}\dot{\bar{\phi}}^2+\left[\frac{5}{\psi^2}\bar{\phi}^2+\frac{1}{2}\bar{\phi}'^2+\frac{2}{\psi}\bar{\phi}\bar{\phi}'\right]_{\psi=\psi_0},&(30)\\ p\equiv\langle -T_j^i\rangle|_{i=j}&=\frac{1}{2}\dot{\bar{\phi}}^2-\left[\frac{5}{\psi^2}\bar{\phi}^2+\frac{1}{2}\bar{\phi}'^2+\frac{2}{\psi}\bar{\phi}\bar{\phi}'\right]_{\psi=\psi_0},&(31)\\ \langle T_\beta^\alpha\rangle|_{\alpha\neq\beta}&=0,&(32)\end{aligned}$$

where dots denote derivative with respect to the time which in our case are zero: $\dot{\bar{\phi}}\Big|_{\psi_0}=0$. Furthermore, from eq. (30) we can make the following identification for the background scalar potential:

$$V[\bar{\phi}]=\left[\frac{5}{\psi^2}\bar{\phi}^2+\frac{1}{2}\bar{\phi}'^2+\frac{2}{\psi}\bar{\phi}\bar{\phi}'\right]_{\psi=\psi_0}.\tag{33}$$

In our model, the hypersurface $\psi=\psi_0$ defines a de Sitter expansion of the universe with a Hubble parameter $H_0=\psi_0^{-1}$. The equation of state for this case is $p=-\rho=-3/\left(8\pi G\psi_0^2\right)$. Then, it is easy to see that the only

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[2] One could consider, for instance, the case when the background field is $\bar{A}^b=\left(\Phi,\bar{A}^1,0,0,0\right)$, that defines an effective homogeneous component of the electric field. However, we would obtain an anisotropic component of the stress tensor T_{10}, which is not compatible with our background, spatially flat, homogeneous and *isotropic* (de Sitter) metric. In general this implies that for the background fields to satisfy Einstein equations, the components $\bar{A}_0;\bar{A}_1;\bar{A}_2;\bar{A}_3$ are highly restricted. In particular we have the following cases to choose:(i)$\bar{A}^i=0$, $\bar{A}^0=\bar{\Phi}(t,\psi_0)$ and $\bar{A}^4=\bar{\phi}(t,\psi_0)$, (ii)$\bar{A}^0=\bar{A}^4=0$ and $\bar{A}^i=\bar{A}_0^i$ constants. In what follows we shall analyze a particular choice of the first case (with $\bar{A}^0=0$), because the other isn't very interesting in the physical sense.

compatible background solution for the field evaluated on the hypersurface is the typical de Sitter solution for a background scalar field: $\bar{\phi}(t,\psi_0)=\bar{\phi}_0$. This means that

$$V[\bar{\phi}_0]=\left[\frac{5}{\psi^2}\bar{\phi}_0^2+\frac{1}{2}\bar{\phi}'^2+\frac{2}{\psi}\bar{\phi}_0\bar{\phi}'\right]_{\psi=\psi_0}=\frac{3H^2}{8\pi G}. \tag{34}$$

A particular solution of (34) is

$$\left(\bar{\phi}'\right)^2\Big|_{\psi=\psi_0=1/H}=\frac{5H^2}{4\pi G}, \tag{35}$$

$$\bar{\phi}'=-5H\bar{\phi}. \tag{36}$$

From eqs. (35) and (36) we obtain

$$\left(\bar{\phi}\right)^2\Big|_{\psi=\psi_0=1/H}=\bar{\phi}_0^2=\frac{1}{20\,\pi\,G}. \tag{37}$$

B. 4D Field fluctuations

Here we consider equations (13), (14) and (15) to search for possible electromagnetic fields generated through this model. In Sect. (IV A) we've seen that the Einstein equations for the background fields exclude any possibility of homogeneous electromagnetic fields.

The equation for the effective scalar $\delta A^4(t,\vec{R},\psi_0)$ on the effective hypersurface (23) is decoupled from the dynamics of the 4-vector. In contrast, the equations for $\delta A^0(t,\vec{R},\psi_0)$ and $\delta A^i(t,\vec{R},\psi_0)$ remain coupled. By the use of our 5D Lorentz gauge evaluated on the foliation $\psi=\psi_0=H_0^{-1}$: $\nabla_a\,A^a|_{\psi_0=H_0^{-1}}=0$, we can express the inhomogeneous term for δA^0 as only a function of δA^4. The solution will involve both, homogeneous and inhomogeneous parts. Once obtained δA^0 and δA^4, we can finally search solutions for the components δA^j. These total solutions are necessary to deduce the effective electric fields. In contrast, as we previously said, the equation of motion for pure magnetic fields may be obtained by just applying the curl in the 3-space to equation (14). The last term in (14) vanishes because is a 3-gradient, and so magnetic fields equations are decoupled. To quantize the field fluctuations on the effective 4D de Sitter spacetime (23), we shall consider the equations (13), (14) and (15), with condition (25), the transformations (22) and the foliation $\psi=\psi_0=1/H_0$. The equal time canonical relations are

$$\left[\delta A_i(t,\vec{R},\psi_0),\Pi^j(t,\vec{R}',\psi_0)\right]\Big|_{\psi_0=1/H_0}=-i\,g_i^j e^{-3H_0t}\,\delta^{(3)}(\vec{R}-\vec{R}'), \tag{38}$$

where g^{ij} are the space-like components of the tensor metric in (23) and $\delta^{(3)}(\vec{R}-\vec{R}')$ is the 3D Dirac's function. Furthermore, the canonical momentum is given by the electric field $\Pi^j\equiv E^j=\nabla^jA^0-\nabla^0A^j$. The equations (13), (14) and (15) with the transformations (22) can be evaluated on the foliation $\psi=\psi_0=1/H_0$ to give the dynamics on the effective 4D spacetime (23). If we take into account the conditions (25), the effective 4D dynamics of the fluctuations describe an effective 4D Lorentz gauge, so that

$$\frac{\partial^2\delta A^0}{\partial t^2}+5H_0\frac{\partial\delta A^0}{\partial t}-H_0^2e^{-2H_0t}\partial_R^2\delta A^0+\nu^2H_0^2\delta A^0=-2H_0^2\frac{\partial\delta\phi}{\partial t}, \tag{39}$$

$$\frac{\partial^2\delta A^j}{\partial t^2}+5H_0\frac{\partial\delta A^j}{\partial t}-H_0^2e^{-2H_0t}\partial_R^2\delta A^j+\nu^2H_0^2\delta A^j=2H_0^2\partial^j\left(\delta A^0+H_0\delta\phi\right), \tag{40}$$

$$\frac{\partial^2\delta\phi}{\partial t^2}+3H_0\frac{\partial\delta\phi}{\partial t}-H_0^2e^{-2H_0t}\partial_R^2\delta\phi+m^2H_0^2\delta\phi=0. \tag{41}$$

describe the 4D dynamics of the fluctuations. A very important fact is that the electromagnetic field fluctuations δA^μ obey a Proca equation with sources. The expansion of the field in temporal modes is

$$\delta A^\mu(t,\vec{R},\psi_0)=\int\frac{d^3K}{(2\pi)^3}\sum_{\lambda=1}^{3}\varepsilon^\mu(\vec{K},\lambda)\left(a_{(\vec{K},\lambda)}e^{-i\vec{K}\cdot\vec{R}}S(K,t,\psi_0)+a^\dagger_{(\vec{K},\lambda)}e^{i\vec{K}\cdot\vec{R}}S^\star(K,t,\psi_0)\right), \tag{42}$$

where $\vec{K}=H_0\,\vec{k}$ (k is a dimensionless wavenumber). Furthermore, $\varepsilon^{\mu}(\vec{k},\lambda)$ are the polarizations [3], such that in the Lorentz gauge the following expression holds:

$$\sum_{\lambda=1}^{3}\varepsilon_{\alpha}(\vec{k},\lambda)\,\varepsilon_{\beta}(\vec{k},\lambda)=-\left(g_{\alpha\beta}-\frac{H_0^2}{m_{eff}^2}k_{\alpha}k_{\beta}\right), \tag{43}$$

where we have introduced the effective mass $m_{eff}^2=H_0^2(\nu^2-\frac{25}{4})$ of the redefined temporal modes $\mathcal{U}_K(t)=e^{5H_0t/2}S(K,t,\psi_0)$, that obey the harmonic equation $\ddot{\mathcal{U}}_K+\omega_K^2(t)\mathcal{U}_K=0$. The time dependent frequency is defined by the relation $K_{\mu}K^{\mu}=m_{eff}^2$.

$$\omega_K^2(t)=\left[m_{eff}^2+(e^{-H_0t}K)^2\right]. \tag{44}$$

Modes with $\omega_K^2>0$ are stable, but those with $\omega_K^2<0$ [i.e., with $k<\left(25/4-\nu^2\right)^{1/2}e^{H_0t}$], are unstable. In the small wavelength limit these behave like plane waves in Minkowski space. Furthermore, the annihilation and creation operators $a_{(K,\lambda)}$ and $a^{\dagger}_{(K,\lambda)}$, comply with the commutation relations

$$\left[a_{(\vec{K},\lambda)},a^{\dagger}_{(\vec{K}',\lambda')}\right]=(2\pi)^3\,g_{\lambda\lambda'}\,\delta^{(3)}(\vec{K}-\vec{K}'). \tag{45}$$

The time dependent modes for the contravariant vector δA^{μ} are

$$S(K,t,\psi_0)=e^{-5H_0t/2}\left\{c_1\mathcal{H}_{\sigma}^{(1)}\left[x(t)\right]+c_2\mathcal{H}_{\sigma}^{(2)}\left[x(t)\right]\right\},\qquad \sigma=\sqrt{\frac{25}{4}-\nu^2},\qquad x(t)=\frac{K}{H_0}\,e^{-H_0t}. \tag{46}$$

We can also obtain the temporal modes for the covariant δA_{μ} which are related to the contravariant ones: $\mathcal{T}_K(t)=e^{2H_0t}\,S(K,t,\psi_0)$. The commutation relations (38) yield the following conditions over these modes

$$\mathcal{T}_K\dot{\mathcal{T}}_K^{\star}-\mathcal{T}_K^{\star}\dot{\mathcal{T}}_K \;=\; -ie^{-H_0t}, \tag{47}$$

$$\mathcal{T}_K\mathcal{T}_K^{\star} \;=\; \frac{e^{-H_0t}}{2w_K(t)}, \tag{48}$$

which are only valid on short wavelength modes for which $\omega_K^2>0$. Equations (47) and (48) give us the normalization conditions for the modes of δA_{μ}. On the other hand, these modes are unstable on cosmological scales: $\omega_K^2<0$, and the expression (47) tends to zero. To apply these conditions we take the small wavelength limit for the Hankel Functions $x(t)\gg|\sigma^2-\frac{1}{4}|$. These means that $K/H_0\,e^{-H_0t}\gg m_{eff}^2$, so that $w_K(t)\simeq Ke^{-H_0t}$. In this limit the conditions (47) and (48) become dependent one of the another. If we choose $c_1=0$, the solution for the modes is

$$\mathcal{T}_K(t)=e^{-\frac{1}{2}H_0t}\sqrt{\frac{\pi}{4H_0}}\,\mathcal{H}_{\sigma}^{(2)}[x(t)], \tag{49}$$

where $\mathcal{H}_{\sigma}^{(2)}[x(t)]$ is the second kind Hankel function.

1. *4D electromagnetic fluctuations*

The electric field for a observer in 4D is defined by its 4-velocity $E_{\nu}=F_{\nu\lambda}u^{\lambda}$. If we choose the particular co-moving frame $u^{\nu}=\left[(H_0\psi_0)^{-1},\vec{0}\right]$, we obtain

$$E_0 \;=\; 0,$$

$$E_i \;=\; \frac{\partial}{\partial X^i}\delta A^0-e^{2H_0t}\frac{\partial}{\partial t}\delta A^i-2H_0\,e^{2H_0t}\delta A^i. \tag{50}$$

[3] parenthesis denotes that sum do no run over these indices.

The magnetic fields are defined by $B_\nu = \frac{1}{2}\epsilon_{\nu\lambda\alpha\beta}u^\lambda F^{\alpha\beta}$, where $\epsilon_{\nu\lambda\alpha\beta} = \sqrt{\left|{}^{(4)}g\right|}\mathcal{A}_{\nu\lambda\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor and $\mathcal{A}_{\nu\lambda\alpha\beta}$ is a totally antisymmetric symbol with $\mathcal{A}_{0123} = -1$. Then for a co-moving observer we will have a magnetic field,

$$\begin{aligned} B_0 &= 0, \\ B_j &= \frac{\sqrt{\left|{}^{(4)}g\right|}}{2}\,\mathcal{A}_{j0kl}\,u^0\,F^{kl}. \end{aligned}$$

From the last expression we can arrive to another that will be useful to obtain an equation of motion for the magnetic fields, we first define the Levi-Civita symbol in the 3-flat space using the co-moving frame: $\epsilon_{jkl} = \mathcal{A}_{j0kl}$ (we note that $\epsilon_{123} = 1$). Hence

$$B_j = \sqrt{\left|{}^{(4)}g\right|}g^{kk'}u^0\epsilon_{jkl}\partial_{k'}A^l. \tag{51}$$

For our particular case we obtain

$$e^{-H_0t}\,B_j = \left[\delta^{kk'}\,\epsilon_{jkl}\,\partial_{k'}\right]A^l. \tag{52}$$

The differential operator between square brackets commutes with the one applied to A^j in the equation (14), so that in the equation of motion for $\mathcal{B}_j = e^{-H_0t}B_j$ there are no sources. We can express the field in Fourier components of the δA^j field

$$\mathcal{B}^j\left(t,\vec{R},\psi_0\right) = \int\frac{d^3K}{(2\pi)^3}\sum_{\lambda=1}^{3}\varepsilon_l(\vec{K},\lambda)\epsilon^{jnl}\left[a_{(\vec{K},\lambda)}\mathcal{V}_n(K,t,\psi_0)\,e^{i\vec{K}\cdot\vec{R}} + a^\dagger_{(\vec{K},\lambda)}\mathcal{V}^\star_n(K,t,\psi_0)\,e^{-i\vec{K}\cdot\vec{R}}\right]. \tag{53}$$

Here $\mathcal{V}_j(K,t,\psi_0) = -iK_j\,S_1(K,t,\psi_0)$ are the temporal modes with their complex conjugate $\mathcal{V}^\star_j(K,t,\psi_0) = iK_j\,S^\star(K,t,\psi_0)$. We perform the vacuum expectation value of the B-fields quadratic amplitude, defined by the invariant product $\langle B^2\rangle \equiv \langle 0|B^\alpha B_\alpha|0\rangle$. For comoving observers $B^0 = 0$ and so we have $B^2 = B^j\,B_j = e^{-2H_0t}\sum_j {B_j}^2 = \sum_j {\mathcal{B}_j}^2$. Then

$$\langle B^2\rangle = \int\frac{d^3K}{(2\pi)^3}(2e^{2H_0t}K^2)S(K,t,\psi_0)S^\star(K,t,\psi_0). \tag{54}$$

We will cut the above integral up to wavelengths that remain well outside the horizon wavenumber $k_H = \frac{5}{2}e^{H_0t}$. In this limit we use the asymptotic limit of the hankel functions for the long wavelength limit $k\,e^{-H_0t} \ll \sqrt{\sigma+1}$. The power spectra is then

$$\mathcal{P}_B(k) = \frac{2^{2\sigma}\Gamma^2(\sigma)H_0^4}{4\pi^3}\,e^{(2\sigma-3)H_0t}\,k^{5-2\sigma}, \tag{55}$$

if we ask for an almost scale invariant spectrum, then $\sigma = \frac{5}{2}+\epsilon$, $\epsilon = -\frac{\nu^2}{5}$ and $\nu^2 \ll 1$. The quadratic amplitude is then

$$\langle B^2\rangle = \frac{45H_0^4}{4\pi^2\nu^2}e^{2H_0t}\left(\frac{5\theta}{2}\right)^{-2\epsilon}, \tag{56}$$

where $\theta \ll 1$ is a control parameter, such that we stay with super Hubble wavelenghts: $k < \theta\,k_H$.

Using the homogeneous solutions of the equations (39), (40) and (41) we can deduce their contribution for electric fields on the infrared (IR) sector, we obtain for comoving observers $\langle E^2\rangle_{IR} = \langle E_A^2 + E_B^2 + E_C^2\rangle_{IR}$, where

$$\langle E_A^2\rangle_{IR} \simeq -H_0^5\frac{e^{-4H_0t}}{\left(\nu^2-\frac{25}{4}\right)}\int_0^{\theta k_H}\frac{dk}{2\pi^2}\,k^6|\mathcal{T}_k|^2, \tag{57}$$

$$\langle E_B^2\rangle_{IR} \simeq -H_0^5\,e^{-2H_0t}\int_0^{\theta k_H}\frac{dk}{2\pi^2}\left(3\,e^{2H_0t}+\frac{H_0^2k^2}{m_{eff}^2}\right)|\dot{\mathcal{T}}_k|^2, \tag{58}$$

$$\langle E_C^2\rangle_{IR} \simeq H_0^5\,e^{-2H_0t}\int_0^{\theta k_H}\frac{dk}{2\pi^2}\sum_j\frac{H_0^2k_0k_j}{m_{eff}^2}(-iH_0k_j)\left(\mathcal{T}_k\dot{\mathcal{T}}_k^\star - \mathcal{T}_k^\star\dot{\mathcal{T}}_k\right). \tag{59}$$

If we choose $\sigma = \frac{5}{2} + \epsilon, \quad \epsilon = -\frac{\nu^2}{5}$ and $\nu^2 \ll 1$, we get

$$\langle E_B^2 \rangle_{IR} \simeq \left(\frac{9}{5\pi}\right)^2 H_0^4 \, e^{2H_0 t} \left[3\theta^{-2} + \frac{4}{25}\theta^{-2\epsilon}\right], \tag{60}$$

$$\langle E_B^2 \rangle_{IR} \simeq 3\left(\frac{9}{5\pi}\right)^2 H_0^4 \, e^{2H_0 t}\theta^{-2}, \tag{61}$$

$$\langle E_C^2 \rangle_{IR} \simeq 0, \tag{62}$$

on cosmological scales. Notice that $\langle E^2 \rangle$ is not scale invariant. Then we can say that on very large scales the amplitude of electromagnetic fields are

$$\left\langle B^2 \right\rangle_{IR}^{1/2} \simeq \frac{3\sqrt{5}}{2\pi\nu} H_0^2 e^{H_0 t} \left(\frac{5\theta}{2}\right)^{\nu^2/5}, \qquad \left\langle E^2 \right\rangle_{IR}^{1/2} \simeq \frac{3^{5/2}}{5\pi} H_0^2 e^{H_0 t}\theta^{-1}. \tag{63}$$

2. *4D inflaton fluctuations*

For the fluctuations of the inflaton field we can make a similar treatment. The Fourier expansion is

$$\delta\phi\left(t, \vec{R}, \psi_0\right) = \int \frac{d^3K}{(2\pi)^3} \left[\alpha_{(\vec{K})}\phi(K, t, \psi_0) \, e^{i\vec{K}\cdot\vec{R}} + \alpha^{\dagger}_{(\vec{K})}\phi^*(K, t, \psi_0) \, e^{-i\vec{K}\cdot\vec{R}}\right], \tag{64}$$

such that the annihilation and creation operators $\alpha_{(K,\lambda)}$ and $\alpha^{\dagger}_{(K,\lambda)}$, comply with the commutation relations

$$\left[\alpha_{(\vec{K})}, \alpha^{\dagger}_{(\vec{K}')}\right] = (2\pi)^3 \; \delta^{(3)}(\vec{K} - \vec{K}'). \tag{65}$$

The solutions for the modes $\phi(K, t, \psi_0)$, are

$$\phi(K, t, \psi_0) = e^{-3H_0 t/2} \; \{c_1 \, J_\mu \, [x(t)] + c_2 \, Y_\mu \, [x(t)]\}, \qquad \mu = \sqrt{\frac{9}{4} - m^2}. \tag{66}$$

The nearly invariant spectrum of the scalar perturbations is obtained for small values of the effective mass: $m \ll 1$. After normalization of the modes, we obtain the standard result(see, for instance[12]) on cosmological scales

$$\left\langle \delta\phi^2 \right\rangle_{IR} \simeq \frac{\Gamma^2(\mu)}{\pi^3(3 - 2\mu)} \left(\frac{2}{\theta\mu}\right)^{2\mu - 3} H_0^2, \tag{67}$$

which is divergent for an exactly scale invariant power spectrum corresponding to a null value of m.

V. FINAL COMMENTS

We have shown how primordial electromagnetic fields and inflaton fluctuations can be generated jointly during inflation using a semiclassical approach to GEMI. The difference with respect other previous works is that, in this one, we have used a Lorentz gauge (rather than a Coulomb gauge). One of the important facts is that our formalism is naturally not conformal invariant on the effective 4D metric (23), which make possible the super adiabatic amplification of the modes of the electromagnetic fields during inflation in a comoving frame on cosmological (super Hubble) scales.

In this letter we have analyzed the simplest nontrivial configuration field: $\bar{A}^b = \left[0, 0, 0, 0, \bar{\phi}(t, \psi_0)\right]$. For these configuration background fields to satisfy the Einstein equations in a de Sitter expansion, the background inflaton field must be a constant on the metric (23): $\bar{\phi}(t, \psi_0) = \bar{\phi}_0$. Then, in the model here developed, the expansion of the universe is driven by the background inflaton field $\bar{\phi}_0$ and background electromagnetic fields are excluded to preserve global isotropy. Notice that back reaction effects are not included in the semiclassical approach here used for the treatment of the Einstein equations. These effects should be included jointly with vectorial metric fluctuations and are the subject of a future work.

To describe the dynamics of the fields, we impose the effective 4D Lorentz gauge ${}^{(4)}\nabla_\mu A^\mu = 0$, given simultaneously by conditions (24) and (25). Therefore, the origin of the generation of the seed of electromagnetic fields and the inflaton field fluctuations during inflation can be jointly studied. The dynamics of δA^μ on the effective 4D metric (23) obey

a Proca equation with sources where the effective mass of the electromagnetic field fluctuations is induced by the foliation $\psi = \psi_0 = 1/H_0$. From the point of view of a relativistic observer this foliation imply that the component of the penta-velocity $U^{\psi} = \frac{d\psi}{dS} = 0$.

Finally, we obtain for small values of the mass ν a nearly scale-invariant long wavelengths power spectrum for $\left\langle B^2 \right\rangle$, which grows as a^2 during inflation. After inflation these fields decreases as a^{-2} to take the present day values on cosmological scales: $\langle B^2 \rangle^{1/2}\big|_{\rm Now} < 10^{-9}$ Gauss[13]. On the other hand, using the homogeneous solutions for $\mathcal{T}_k(t)$, we obtained that $\left\langle E^2 \right\rangle_{IR}$ also grows as a^2 but has a scale dependent power spectrum that goes as $\mathcal{A}_1(t)\, k^2 + \mathcal{A}_2(t)\, k^{-2}$; the first term means that (for a given time), electric fields become more important on shorten scales and the second one become more important on very large scales. In what respect to the inflaton field fluctuations $\left\langle \delta\phi^2 \right\rangle$, they are scale invariant on cosmological scales, but the amplitude is freezed in agreement with the predictions of standard 4D inflation.

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